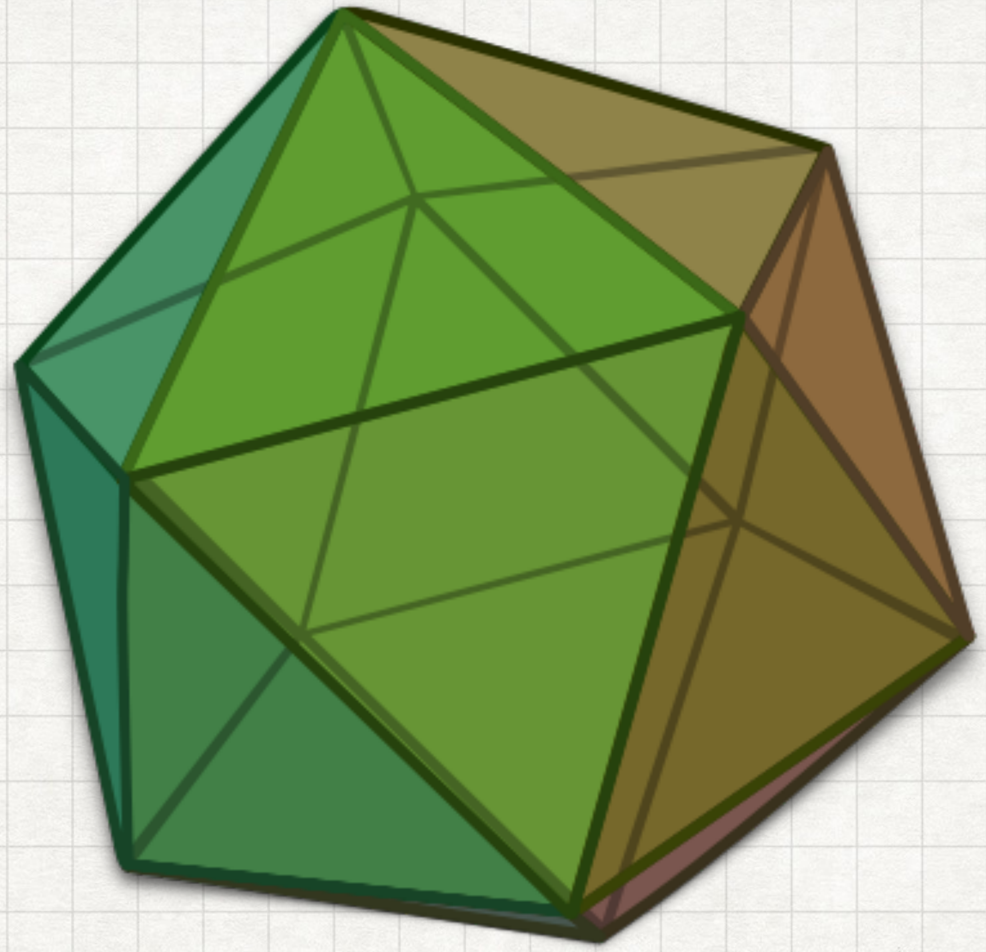

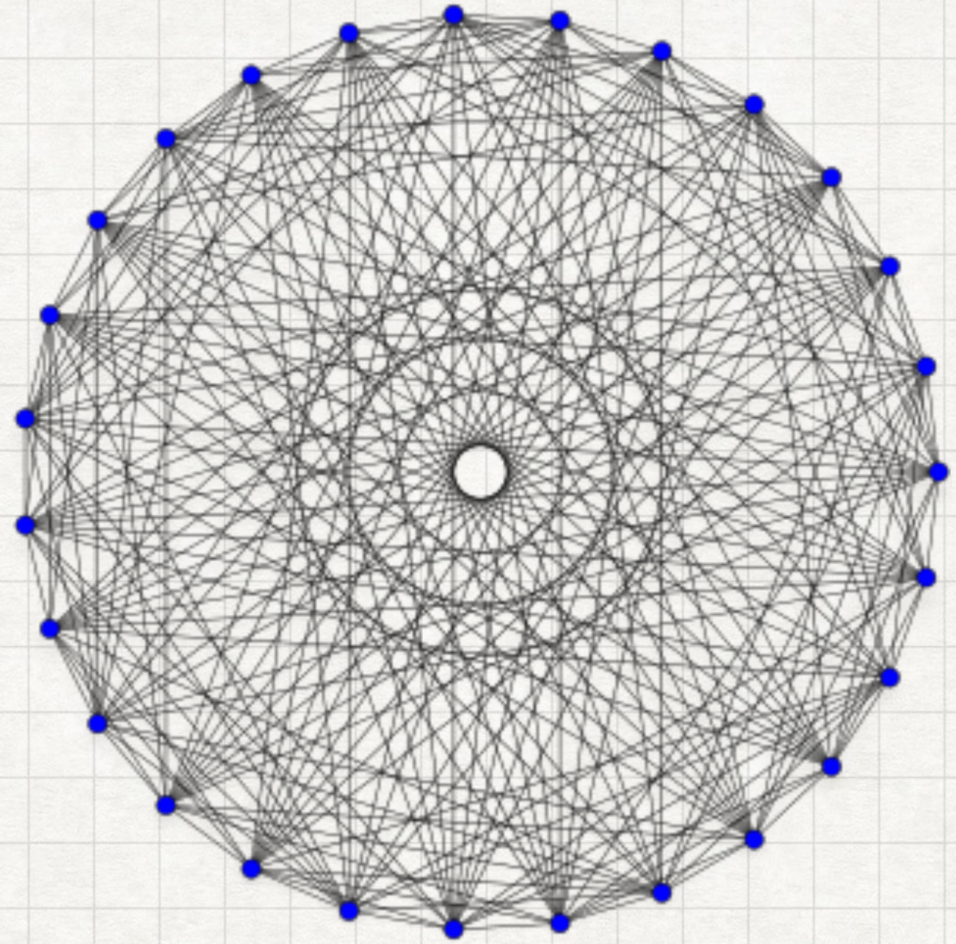

STATE OF THE ART ON EQUIANGULAR LINES

By: Igor Balla

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Connections:

- Elliptic geometry
- Frame theory
- Theory of polytopes
- Banach space theory
- Spectral graph theory
- Algebraic number theory
- Quantum information theory

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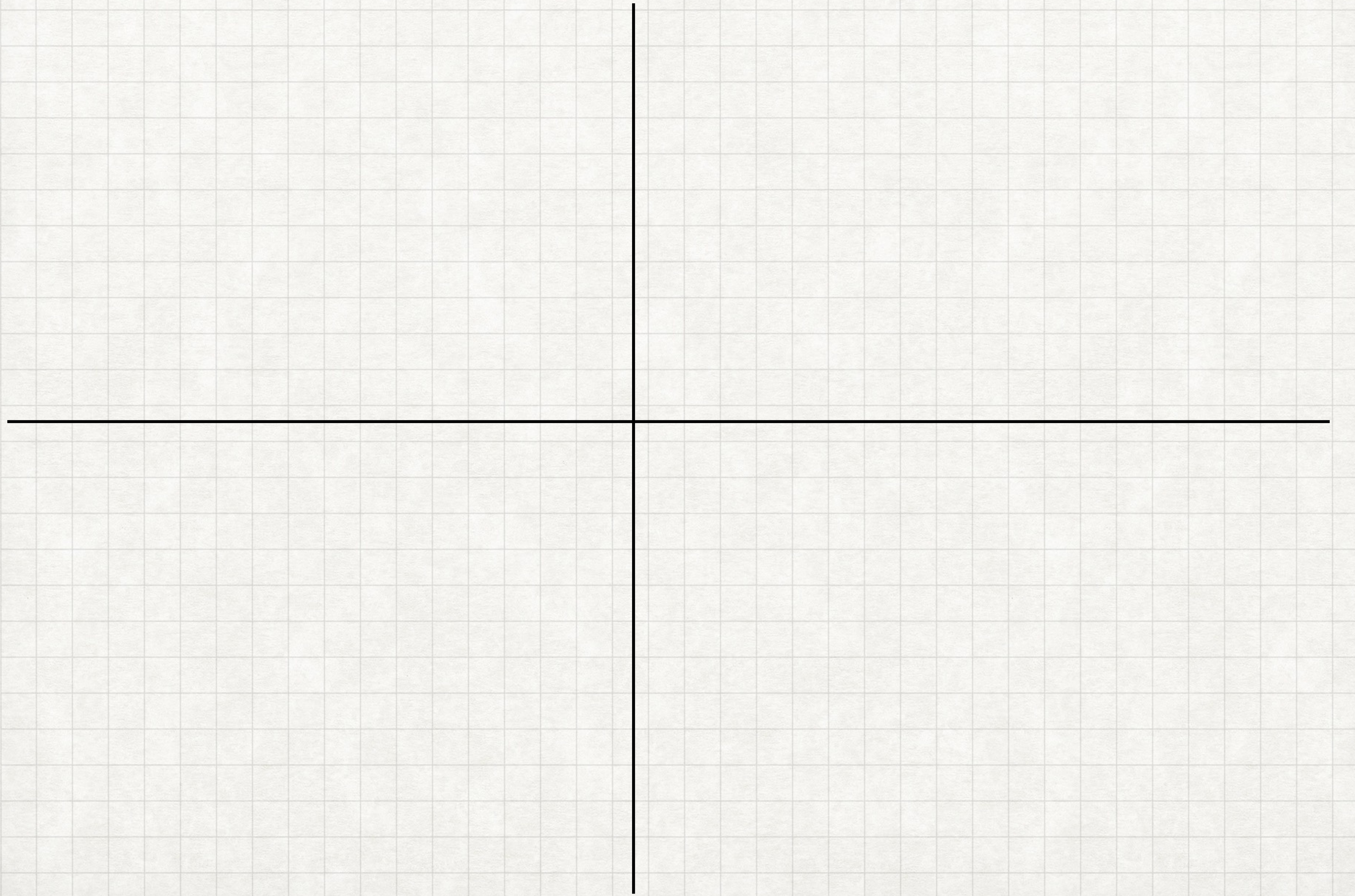
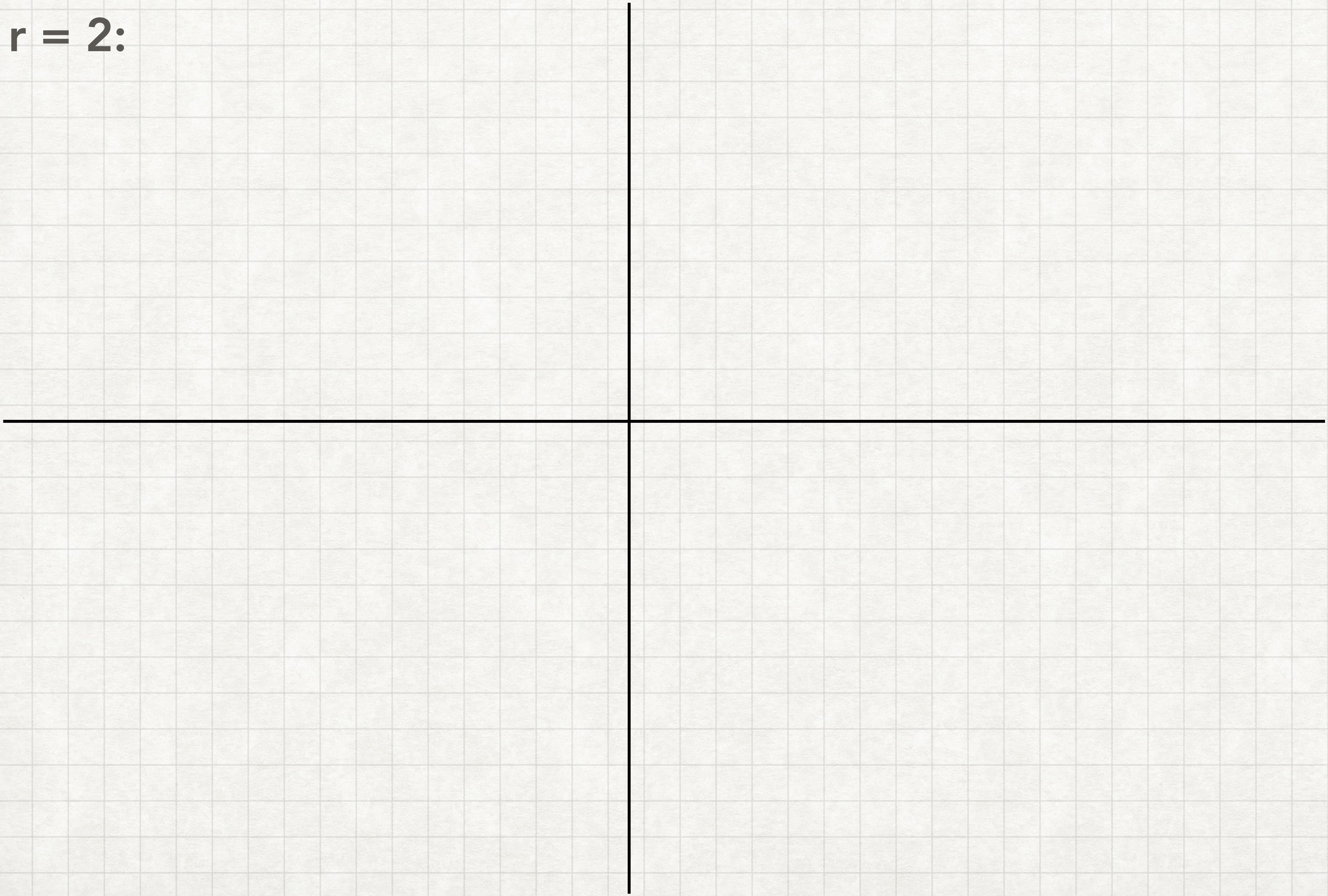
Examples



Examples

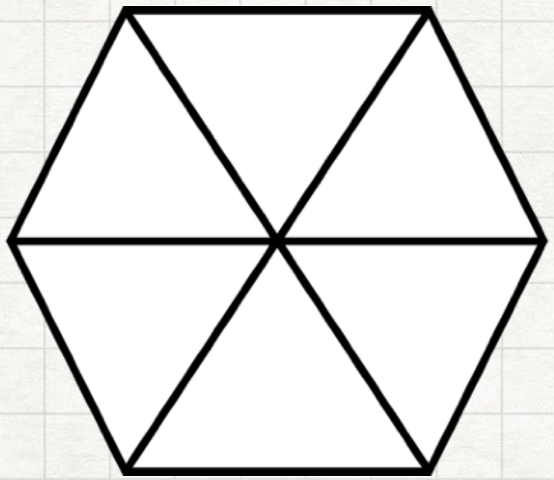
$r = 2$:



Examples

r = 2: Regular Hexagon

3 lines



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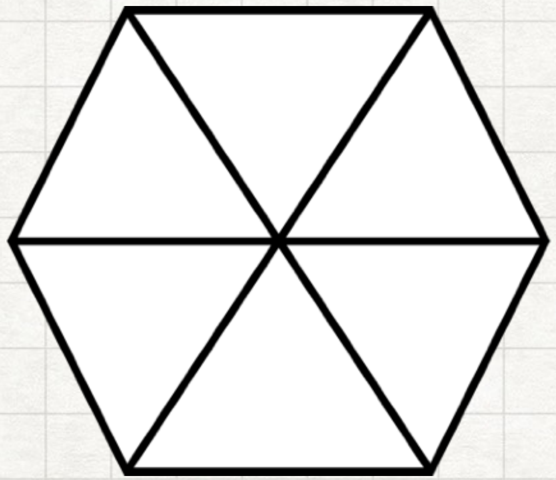

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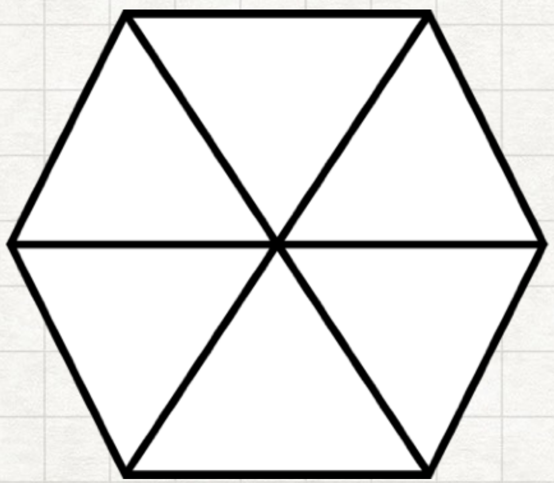
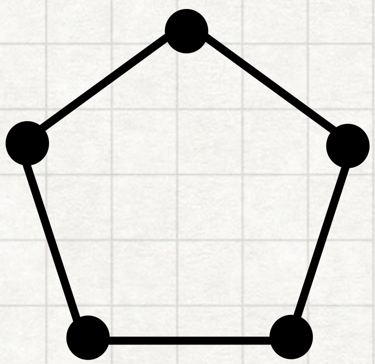
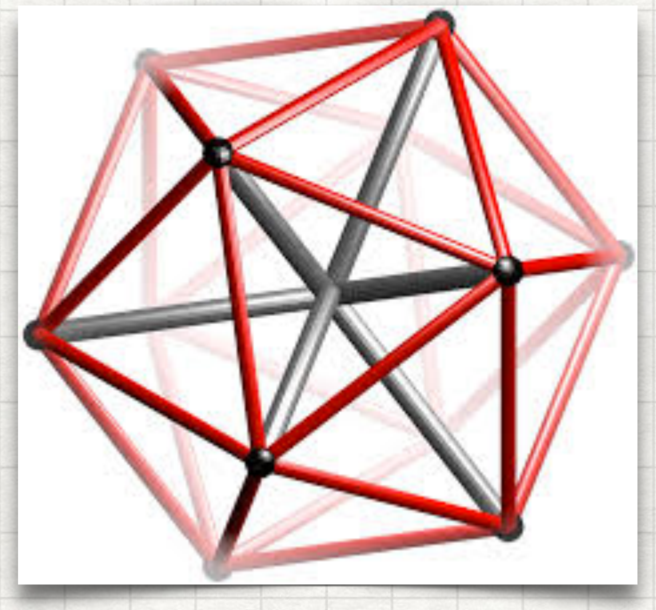


r = 3: Regular Icosahedron

6 lines





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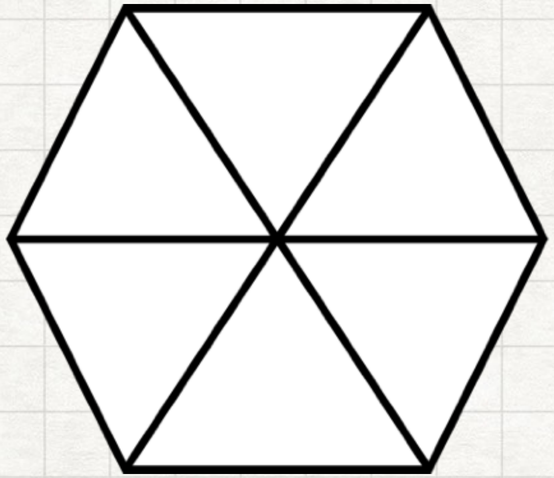

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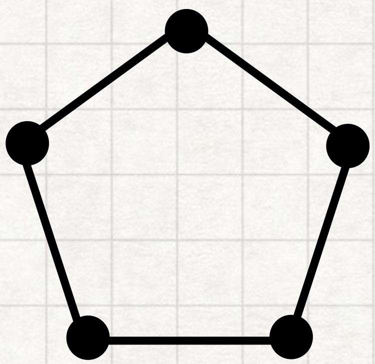

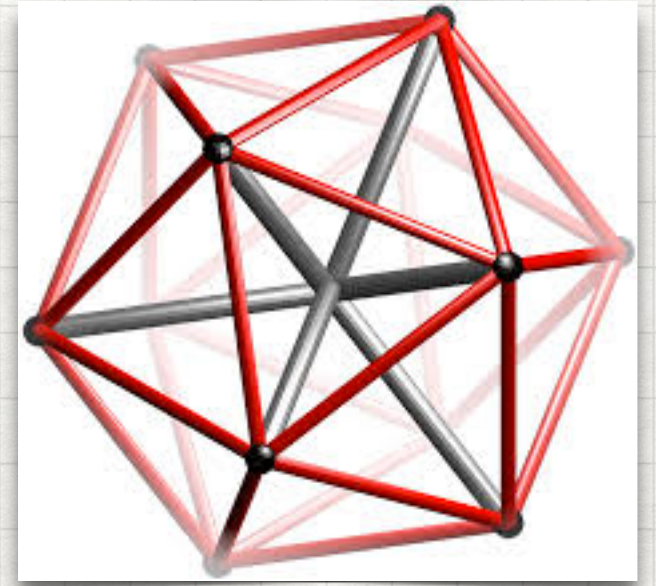

r = 7:

Examples

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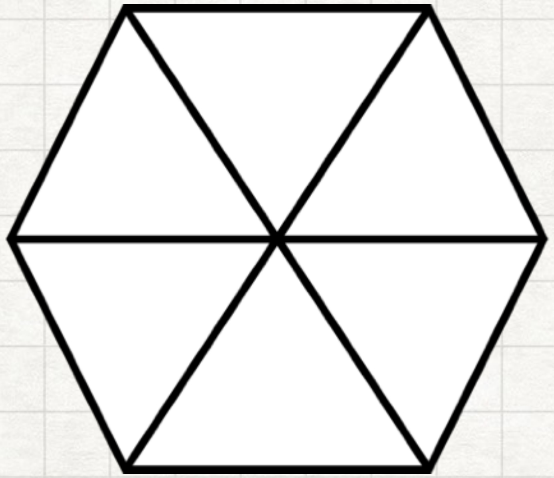

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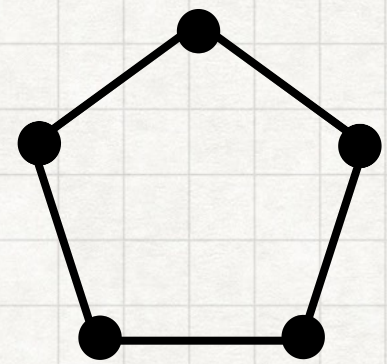
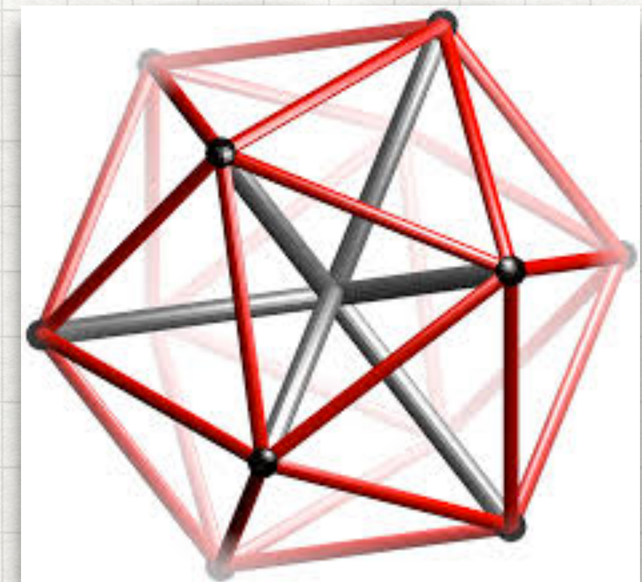
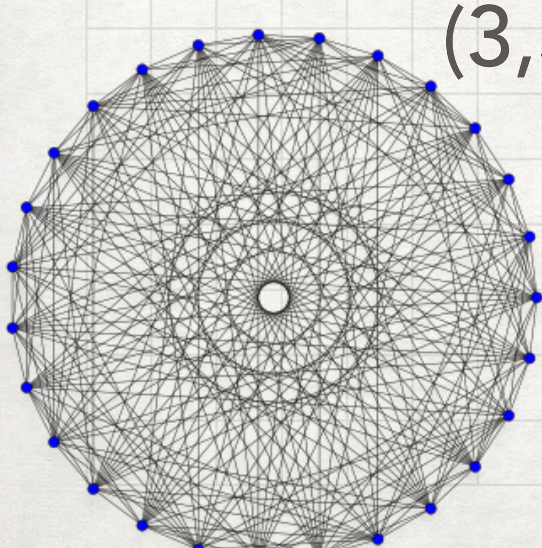




r = 7:

28 lines

Take all 28 permutations of the vector (3,3,-1,-1,-1,-1,-1,-1).



Schläfli Graph (E8 lattice)

Examples

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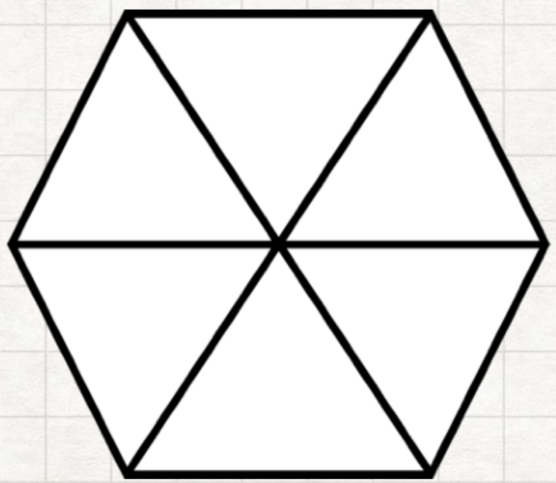

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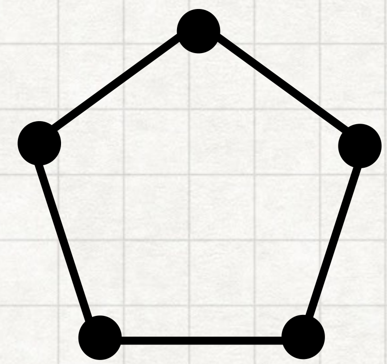

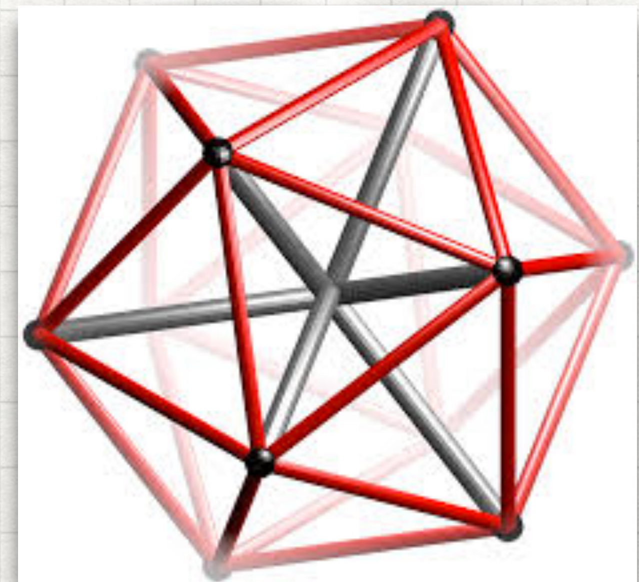

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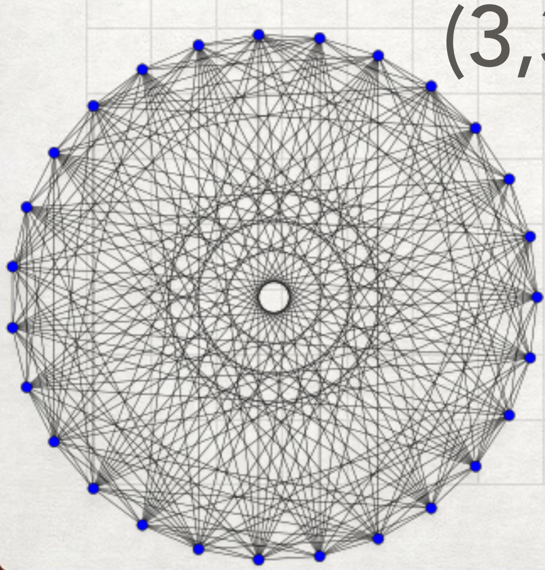


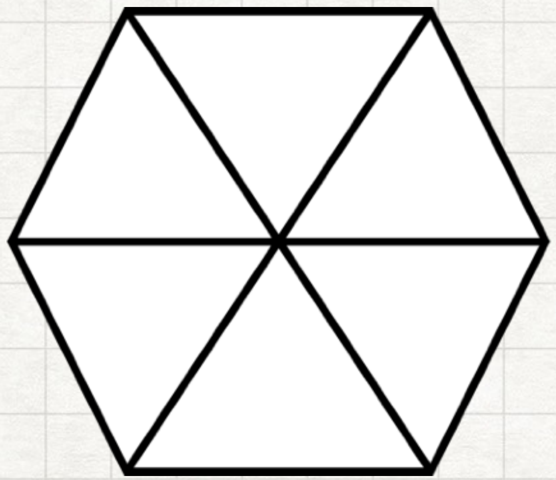
Schläfli Graph (E8 lattice)

r = 23:

Examples

r = 2: Regular Hexagon

3 lines



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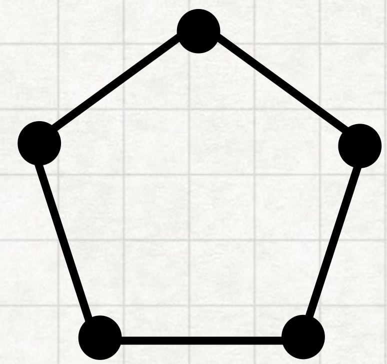

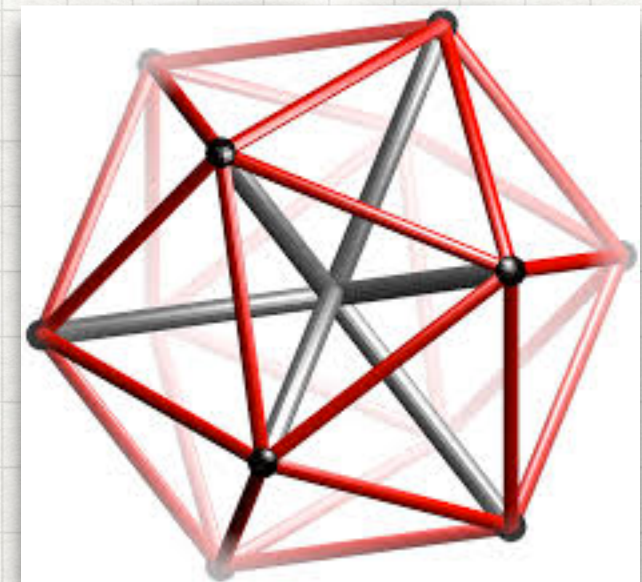

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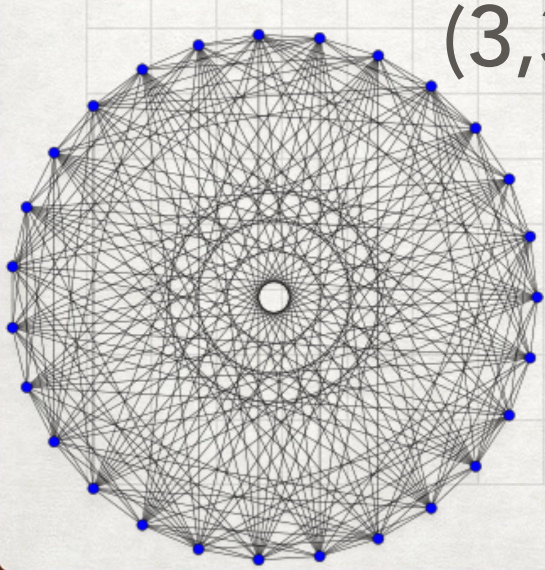

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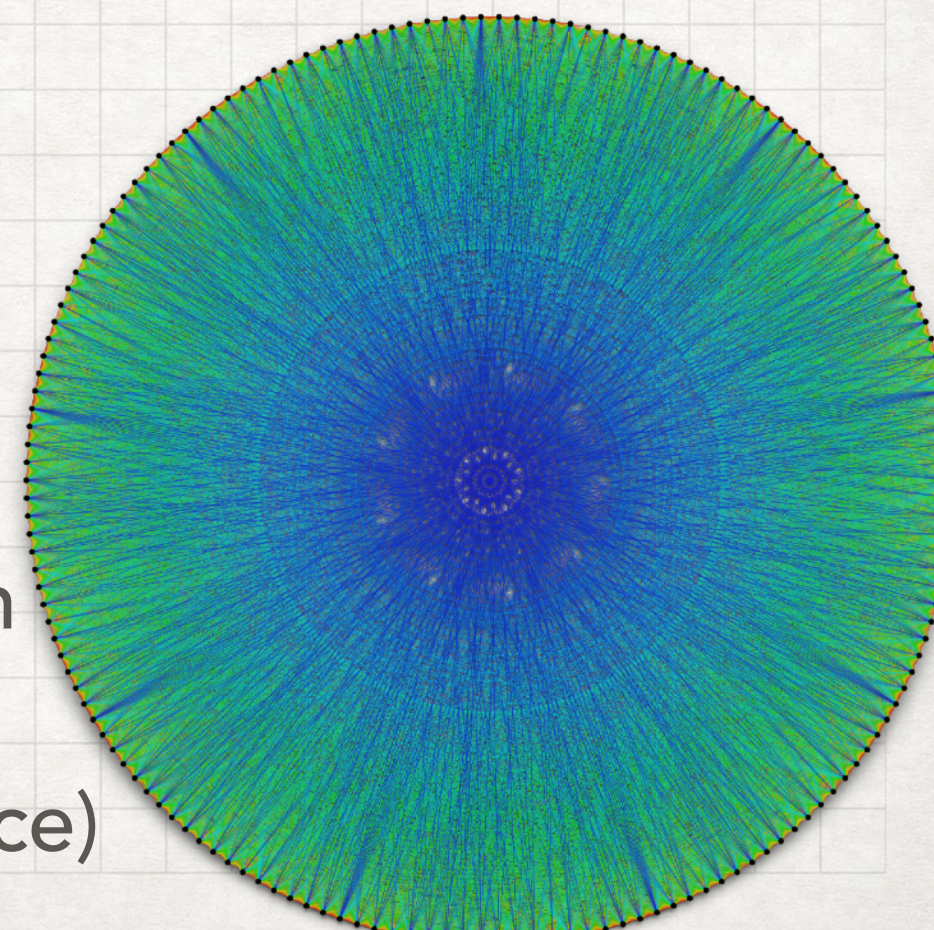
276 lines

McLaughlin Graph (Leech lattice)



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Hence they are linearly independent. □

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Theorem[Relative Bound] (Lemmens, Seidel 73): $N_\alpha(r) \leq r\frac{1-\alpha^2}{1-r\alpha^2}$ for all $r \leq 1/\alpha^2 - 2$.

Recent progress

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For $\frac{1}{\alpha^2} - 2 \leq r \lesssim \frac{1}{4\alpha^4}$, equality occurs if and only if the absolute bound is tight in dimension $\frac{1}{\alpha^2} - 2$

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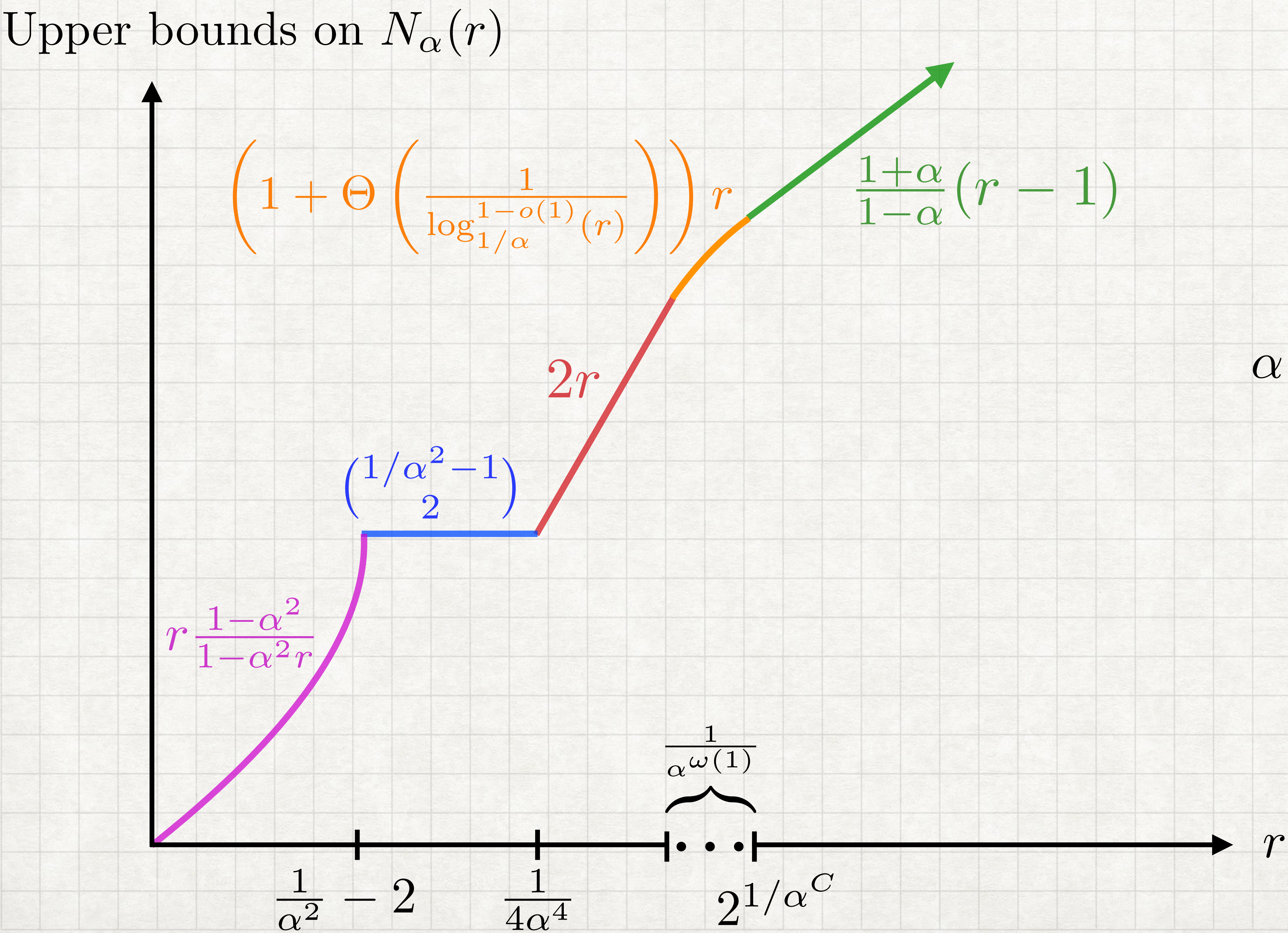
$$N_{\frac{1}{2k-1}}(r) = \left\lfloor \frac{r-1}{1-1/k} \right\rfloor.$$

Recent progress



$\alpha = \frac{1}{2k-1}$

$k \in \mathbb{N}$

Connection to spectral graph theory

Given a family of n equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, we can pick a unit vector along each line to get vectors $v_1, \ldots, v_n$ satisfying $\langle v_i, v_j \rangle = \pm\alpha$ for all $i \neq j$.

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The adjacency matrix A of this graph satisfies

$$M = (1 - \alpha)I + \alpha J - 2\alpha A.$$

Connection to spectral graph theory

Given a family of n equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, we can pick a unit vector along each line to get vectors $v_1, \ldots, v_n$ satisfying $\langle v_i, v_j \rangle = \pm\alpha$ for all $i \neq j$.

The Gram matrix M of these vectors, defined by $M_{i,j} = \langle v_i, v_j \rangle$, is $n \times n$, positive semidefinite, and has rank at most r.

Consider the graph G with vertex set $v_1, \ldots, v_n$ such that $v_i v_j \in E(G)$ if and only if $\langle v_i, v_j \rangle = -\alpha$.

The adjacency matrix A of this graph satisfies

$$M = (1-\alpha)I + \alpha J - 2\alpha A.$$

If $n \geq r+2$, then its second largest eigenvalue is $\lambda_2 = \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)$ and has multiplicity at least $n - r - 1$.

New bounds for the second eigenvalue of a graph

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Theorem(B. 21-25)**:** Let G be a d-regular graph on n vertices with second eigenvalue λ_2 and let $\varepsilon > 0$ be fixed. Then

$$\lambda_2 \geq \begin{cases} \Omega\left(\sqrt{d}\right) & \text{if } 1 \leq d \leq n^{2/3} \\ \Omega\left(n/d\right) & \text{if } n^{2/3} < d \leq n^{3/4} \\ \Omega\left(d^{1/3}\right) & \text{if } n^{3/4} < d \leq (1/2 - \varepsilon)n. \end{cases}$$

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Theorem(B., Bucić 24): Let G be a graph n on vertices with second eigenvalue $\lambda_2 > 0$ and maximum degree $\Delta \geq 2$. Then the multiplicity of λ_2 satisfies

$$m(\lambda_2) \leq \max\left\{\frac{n}{\lambda_2^{1-o(1)}}, \frac{n}{(\log_\Delta n)^{1-o(1)}}\right\}.$$

Moreover, if $n \geq 2^{\Delta^{\Omega(1)}}$, then $m(\lambda_2) \leq \frac{n}{\lambda_2+1} + n^{o(1)}$.

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Proof sketch: Start with the Gram matrix M of the unit vectors $v_1, \ldots, v_n$ spanning n lines. Consider its largest eigenvalue λ_1.

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If $\lambda_1 \le \frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right)$, then the first bound $n \le \binom{1/\alpha^2-1}{2}$ follows immediately from $n(1+\alpha^2(n-1)) = \mathrm{tr}(M^2) \le \lambda_1 \mathrm{tr}(M) = \lambda_1 n$.

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The second bound $n \leq (2+o(1))r$ then follows by applying the inequality $\mathrm{tr}(H)^2 \leq \mathrm{rk}(H)\mathrm{tr}(H^2)$ with $H = M - \alpha J$. $\square$

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Collections of r^2 complex equiangular lines in $\mathbb{C}^r$ are known as SIC-POVMs/SICs in quantum information theory.

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Theorem(B.)**:** If $r \leq \frac{1-o(1)}{\alpha^3}$, then $N_\alpha^{\mathbb{C}}(r) \leq \left(\frac{1}{\alpha^2} - 1\right)^2$, with equality if and only if there exists a SIC in $1/\alpha^2 - 1$ dimensions.

Otherwise $N_\alpha^{\mathbb{C}}(r) \leq \frac{1+\alpha}{\alpha} r + O\left(\frac{1}{\alpha^3}\right)\,.$

Future directions for research

- Unit vectors corresponding to equiangular lines are equivalently spherical $\{\alpha, -\alpha\}$-codes. Extend methods to more general spherical L-codes.

- Determine $N_\alpha^{\mathbb{C}}(r)$.

- Generalize to other graph matrices (ex: Laplacian).

- Generalize to equiangular subspaces.

- Is there a constant $C > 0$ such that if $n \geq \Delta^C$, then any graph with n vertices and max degree Δ has second eigenvalue multiplicity $m(\lambda_2) \lesssim \frac{n}{\lambda_2}$?

thank you
рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
mersi
kia ora
barka
welalin
tack
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tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
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grazzi
gracias
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asante
manana
obrigada
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mercé
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তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci
תודה רבה